63-3-4

THE UNIVERSITY OF WISCONSIN
madison, wisconsin

UNITED STATES ARMY

MATHEMATICS RESEARCH CENTER

MATHEMATICS RESEARCH CENTER, UNITED STATES ARMY
THE UNIVERSITY OF WISCONSIN

Contract No.: DA-11-022-ORD-2059

POWER SERIES WHOSE PARTIAL SUMS HAVE FEW ZEROS IN AN ANGLE

J. Korevaar and T. L. McCoy

MRC Technical Summary Report #387
April 1963

Madison, Wisconsin

ABSTRACT

Let $\sum a_n z^n$ be a power series different from a polynomial, $s_n(z)$ its partial sum of order n. Let $\nu_n(\delta)$ denote the minimum of the number of zeros of s_n in any angle of opening δ and vertex O. It has been known for several years that if $\nu_n(\delta) = o(n)$ for some $\delta > 0$, then the power series must represent an entire function of order zero. In the present paper it is assumed that $\nu_n(\delta) \leq An^{\alpha}$, $n = 1, 2, \ldots$ for some $\delta > 0$ and $0 \leq \alpha < 1$. A harmonic measure technique is used to estimate the growth of the entire function in this case. Taking $A \geq 1$ when $\alpha = 0$ it is shown that $a_n = O\{\exp(-\epsilon n^{2-\alpha})\}$, with $\epsilon \geq 1/\Omega A$ and $\log_{10}\Omega = 10^{100/\delta}$. Apart from the value of ϵ this estimate for a_n is best possible. The proof shows also that if $a_n \neq 0$ and $n \geq n_0$ there is a coefficient $a_{n-p} \neq 0$ with $p < \Omega A n^{\alpha}$. Thus in the case of a zero free angle the power series can have no unbounded gaps, and $a_n = O\{\exp(-\epsilon n^2)\}$.

POWER SERIES WHOSE PARTIAL SUMS HAVE FEW ZEROS IN AN ANGLE

J. Korevaar and T. L. McCoy*

1. Introduction. Let

$$\sum_{0}^{\infty} a_n z^n \tag{1.1}$$

be a formal power series different from a polynomial,

$$s_n(z) = \sum_{0}^{n} a_k z^k \tag{1.2}$$

its partial sum of order n. We denote by

$$\nu_n(\delta) \tag{1.3}$$

the minimum of the number of zeros of s_n in any angle of opening δ and vertex O.

If for some $\delta > 0$

$$\nu_n(\delta) = o(n) \quad \text{as} \quad n \to \infty, \tag{1.4}$$

then it follows from the work of Jentzsch [6], Carlson [1, 2], Rosenbloom

*Illinois Institute of Technology, Chicago, Illinois

Sponsored by the Mathematics Research Center, U. S. Army, Madison, Wisconsin under Contract No. DA-11-022-ORD-2059.

[9, 10] and the first author [7] that the series (1.1) represents an entire function of order zero. However, until very recently it was not known what the implications are of restrictions stronger than (1.4). In this paper we assume that for some $\delta > 0$ and an α with $0 \leq \alpha < 1$,

$$(1.5) \qquad \nu_n(\delta) \leq An^{\alpha}, \quad n = 1, 2, \ldots \quad .$$

Not long ago Hedstrom and the first author [5] made a very detailed study of the zeros of the partial sums of the special series $\sum_0^\infty \exp(-n^\beta) z^n$, with $1 < \beta < 2$. They found that in this case $\nu_n(\delta) \sim c(\delta, \beta) n^{2-\beta}$, and conjectured that this is about as small as the $\nu_n(\delta)$ can be for power series with coefficients of comparable size.

However, it took the ingenuity and the powerful techniques of Ganelius [4] to prove, during a 1962 analysis conference at Wisconsin, that (1.5) implies an estimate of the form

$$(1.6) \qquad a_n = O\{\exp(-\epsilon\, n^{2-\alpha})\} \qquad (\epsilon > 0) \quad .$$

In particular, if every partial sum s_n has a zero free angle of fixed opening $\delta > 0$ and vertex O, then $a_n = O\{\exp(-\epsilon\, n^2)\}$, a result heretofore known only in a few special cases [3, 8].

In the present note we combine Ganelius' ideas with a harmonic measure technique used in the second author's (unpublished) Ph. D. thesis [8]. We thus obtain a somewhat more transparent proof of (1.5)$\rightarrow$(1.6), and are able to estimate ϵ in terms of A and δ (taking $A \geq 1$ when $\alpha = 0$):

$$(1.7) \qquad \epsilon \geq \frac{1}{\Omega A}, \quad \text{with} \quad \log_{10}\Omega = 10^{100/\delta} .$$

(Ganelius' proof, which makes use of Vitali's theorem, does not give an explicit bound for ϵ).

Our proof also shows that if a_k is a rather large coefficient (that is, a coefficient corresponding to a vertex of the Newton polygon), then there is a rather large coefficient a_{k-p} such that $0 < p < \Omega A k^{\alpha}$ whenever k is sufficiently large. In particular, if the partial sums have a zero free angle of fixed opening and vertex O the power series can not have unbounded gaps.

2. Outline of the proof (special case).

We set

$$(2.1) \qquad |a_n| = \exp\{-f(n)\} ,$$

and assume in this section that $f(x)$ resembles $x^{2-\alpha}$ to the extent that

$$(2.2) \qquad f \in C^2, \quad 0 < f''(x)\downarrow , \quad xf''(x)\uparrow\infty .$$

Since

$$f(n-k) = f(n) - kf'(n) + \frac{1}{2}k^2 f''(n-\theta k), \quad 0<\theta<1 ,$$

with small f'', we write for fixed n

$$s_n(z) = \sum_{k=0}^{n} \exp\{-f(n-k) + i\beta_k\} z^{n-k}$$

$$= z^n e^{-f(n)} \sum_{k=0}^{n} b_k \{e^{f'(n)}/z\}^k ,$$

where

(2.3) $$b_k = \exp\{-f(n-k) + f(n) - kf'(n) + i\beta_k\} .$$

We now consider the auxiliary polynomial

(2.4) $$S(z) = \sum_0^n b_k z^k = e^{f(n)-nf'(n)} z^n s_n(e^{f'(n)}/z) .$$

This polynomial has the same minimum number of zeros in an angle of opening δ and vertex O as s_n; by rotation we may assume that the number of zeros of S in the angle $|\arg z| < \frac{1}{2}\delta$ is bounded by An^α.

The coefficients of S are of course much more tractable than those of s_n. By our assumptions $f''(n-\theta k) \geq f''(n)$ for all k, and $f''(n-\theta k) \leq f''(\frac{1}{2}n) \leq 2f''(n)$ at least for $k \leq \frac{1}{2}n$. Thus, setting $f''(n) = \lambda$,

$$|b_k| = \exp[-\tfrac{1}{2}k^2 f''(n-\theta k)] \begin{cases} \leq \exp(-\tfrac{1}{2}\lambda k^2) \leq 1 & \text{for all } k \leq n\ , \\ \geq \exp(-\lambda k^2) & \text{for } k \leq \tfrac{1}{2}n\ . \end{cases} \tag{2.5}$$

Besides S we consider a polynomial T obtained by removal of the zeros $z_1, \ldots, z_N$ in a sufficiently large sector $|\arg z| < \frac{1}{2}\delta$, $|z| < R$:

$$T(z) = S(z) / \prod_1^N (z - z_j)\ . \tag{2.6}$$

From (2.5) it is easy to obtain an upper bound for $\log|S(z)|$ in terms of λ, and hence an upper bound for $\log|T(z)|$ in terms of λ and N.

We then introduce a holomorphic branch of $\log T(z)$ in the sector $|\arg z| < \frac{1}{2}\delta$, $|z| < R$. From the known bound on its real part and a bound at $z = 1/3$ we immediately obtain a bound on $|\log T(z)|$ in terms of λ, N and δ which is valid throughout the smaller region $|\arg z| \leq \frac{1}{4}\delta$, $1/3 \leq |z| \leq \frac{1}{2}R$.

Next we apply a harmonic measure argument to $\log|\log T(z)|$ in the smaller region. Using the relative smallness of the function on the arc $|z| = 1/3$ we obtain an estimate for $|\log T(x)|$ on a segment of the positive real axis. From this we obtain an estimate for $\log|S(x)|$ in terms of λ, N and δ on the segment $0 \leq x \leq R/4$.

We finally apply a harmonic measure argument to $\log|S(z)|$ in the domain bounded by the circle $|z| = R/4$ and the segment $0 \leq x \leq R/4$ of the real axis. Using our estimates on the two parts of the boundary we obtain an improved estimate for $\log|S(z)|$ on circles of moderate size.

The latter estimate gives a new upper bound for $|b_k z^k|$ which depends on λ in such a way that comparison with the lower bound known from (2.5) leads to a lower bound for $\lambda = f''(n)$, and hence for $f(n)$.

3. The general case. In the case of "arbitrary" coefficients a_n the beginning of the proof has to be refined. We may of course assume that $\sum a_n z^n$ represents an entire function (so that $(1/n) \log|1/a_n| \to \infty$), and not a polynomial. We now introduce the Newton polygon g, that is, the maximal convex minorant of the function $f = \log|1/a|$. We have $f \geq g$, and $f(n) = g(n)$ for every n which corresponds to a vertex of the polygon; since $f(n)/n \to \infty$ there are infinitely many vertices.

The derivative g' will be piecewise constant and non-decreasing; we define $g'(n) = g'(n-)$. We note that $g'(x) \uparrow \infty$ (or else $g(x) = O(x)$ and hence $f(n) = O(n)$ on the sequence of vertex indices n).

From here on n will always correspond to a vertex of the Newton polygon. Using an idea of Ganelius [4] we define p_n as the smallest positive integer such that

$$g'(n) - g'(n - p_n) \geq 1 \qquad (n \geq n_1) ; \tag{3.1}$$

$n - p_n$ <u>will also be a vertex index.</u> We remark that $1/p_n$ corresponds to the quantity $f''(n)$ in Section 2.

<u>Lemma 1</u>. <u>If</u>

$$p_n \leq Cn^{\alpha} \qquad (0 \leq \alpha < 1) \tag{3.2}$$

<u>for all vertex indices</u> $n \geq n_2$, <u>then</u>

$$f(x) \geq g(x) \geq x^{2-\alpha}/6C \quad (x \geq x_1) . \tag{3.3}$$

<u>Proof.</u> Set $p_n^{(1)} = p_n$, and let $p_n^{(i)}$ be the p_m which corresponds to the vertex index $m = n - p_n^{(1)} - \dots - p_n^{(i-1)}$. Then

$$g'(n) - g'(n - p_n^{(1)} - \dots - p_n^{(j)}) \geq j ,$$

hence if $j = [n^{1-\alpha}/2C]$ and $n \geq 2n_2$,

$$g'(n) \geq g'(\tfrac{1}{2}n) + [n^{1-\alpha}/2C] \geq n^{1-\alpha}/2C \quad (n \geq n_3) .$$

Now $g'(x) \geq g'(n - p_n)$ for $n - p_n < x \leq n$, and thus

$$g'(x) \geq x^{1-\alpha}/3C \qquad (x \geq x_2) ;$$

integration gives (3.3).

We conclude from this lemma that if

$$\mu_n = p_n/n^\alpha \to 0$$

there is nothing to prove. <u>We assume therefore that</u>

$$\limsup (\mu_n = p_n/n^\alpha) = K > 0$$

(which is always true when $\alpha = 0$). <u>If</u> K <u>is infinite we will only look at those vertex indices</u> n <u>for which</u> $\mu_n \geq \mu_k$ <u>for all vertex indices</u> $k < n$. <u>If</u> K <u>is finite we restrict ourselves to those</u> n <u>for which</u> $\mu_n > 3K/4$. In both cases there will be an integer $n_0 \geq 0$ such that

$$\mu_n \geq (2/3)\,\mu_k \tag{3.4}$$

whenever the special vertex index n is $\geq n_0$ and $n_0 \leq k \leq n$. It follows that <u>for our sequence of special vertex indices</u> n,

$$p = p_n \geq (2/3)p_k \quad \text{whenever} \quad n_0 \leq k \leq n\,. \tag{3.5}$$

Using the notation of the above proof we will have

$$g'(n) - g'(n-t) \geq j \quad \text{whenever} \quad t \geq p_n^{(1)} + \dots + p_n^{(j)}\,.$$

By (3.5) the inequality for t is certainly satisfied if $t \geq (3/2)pj$ or $j \leq 2t/3p$ (and $n - t \geq n_0$), hence

(3.6) $$g'(n) - g'(n-t) \geq [2t/3p] \quad \text{whenever} \quad t \leq n - n_0 .$$

We now introduce the auxiliary polynomial

(3.7) $$S(z) = \sum_0^n b_k z^k = e^{g(n)-ng'(n)} z^n s_n(e^{g'(n)}/z)$$

where n is a special vertex index; this time

(3.8) $$b_k = \exp\{-f(n-k) + g(n) - kg'(n) + i\beta_k\} .$$

As before we may assume that the number of zeros of S in the angle $|\arg z| < \frac{1}{2}\delta$ is bounded by An^{α}. Note that $|b_0| = 1$; dividing $S(z)$ by b_0 we may assume that $b_0 = 1$.

Since $f \geq g$ and g is convex,

(3.9) $$|b_k| \leq \exp\{-g(n-k) + g(n) - kg'(n)\} \leq 1 \quad \text{for all} \quad k \leq n .$$

Setting $1/p = \lambda$ we have by (3.6)

$$g(n-k) - g(n) + kg'(n) = \int_0^k \{g'(n) - g'(n-t)\}\,dt$$

$$\geq \int_0^k [2\lambda t/3]\,dt \geq \int_0^k (2\lambda t/3 - 1)\,dt = (\lambda/3)k^2 - k$$

provided $k \leq n - n_0$. For $k > n - n_0$ we have the same lower bound as for $k = n - n_0$, hence a short computation shows that we can use the lower bound $\frac{1}{4}\lambda k^2 - k$ for all $k \leq n$ provided we take $n \geq 16 n_0$, say. On the other hand, since $g'(n) - g'(n-t) < 1$ for $t < p$,

$$f(n-p) - g(n) + pg'(n) = g(n-p) - g(n) + pg'(n)$$

$$= \int_0^p \{g'(n) - g'(n-t)\}\,dt < p .$$

Thus for $n \geq 16 n_0$,

$$(3.10) \quad \begin{cases} |b_k| \leq \exp(-\frac{1}{4}\lambda k^2 + k) & \text{for all } k \leq n , \\ |b_p| \geq e^{-p} . & \end{cases}$$

From here on the proof goes as in Section 2 ; we will turn to the details after we formulate some auxiliary results.

4. Two lemmas for angular regions. We first prove an analog of the Borel-Carathéodory inequality which can be used in a sector.

Lemma 2. Suppose that F is holomorphic in the sector

$$|\arg z| < \frac{1}{2}\gamma \leq \pi, \quad 0 < |z| < 2^{\gamma/\pi} r ,$$

and that

$$\operatorname{Re} F(z) \leq A$$

there. Then in the smaller region

$$|\arg z| \leq \frac{1}{4}\gamma, \qquad 0 < a \leq |z| \leq r$$

one has the inequality

$$|F(z)| \leq |F(a)| + 8\{A + |F(a)|\}(r/a)^{\pi/\gamma},$$

provided r/a is sufficiently large ($r/a \geq 30^{\gamma/\pi}$ will do).

It will be sufficient to sketch the proof for the case $\gamma = \pi$. We proceed as in [11, Section 5.5]. Assume that $F(a) = 0$. We may then assume that $A > 0$. It follows that the function

$$G(z) = \frac{F(z)}{2A - F(z)}$$

will be holomorphic for $|\arg z| < \frac{1}{2}\pi$, $0 < |z| < 2r$, and bounded by 1. Hence since it vanishes for $z = a$, the maximum modulus theorem shows that in this sector

$$\left|G(z)\,\frac{z+a}{z-a}\right| \leq \frac{2r+a}{2r-a},$$

provided $r/a \geq \frac{1}{2}$. Thus in the region $|\arg z| \leq \frac{1}{4}\pi$, $a \leq |z| \leq r$, where $|z-a|/|z+a|$ is maximal at $z = re^{i\pi/4}$,

$$|G(z)| \leq \left|\frac{re^{i\pi/4} - a}{re^{i\pi/4} + a}\right| \frac{2r+a}{2r-a},$$

and if $r/a \geq 30$ the right hand side is certainly $< 1 - a/4r$.

Expressing F in terms of G we find that

$$|F(z)| \leq Ar/a \ .$$

The general case easily follows by applying the preceding to $F(z) - F(a)$.

We next estimate some harmonic measures.

Lemma 3. *Let* D *be the domain*

$$|\arg z| < \frac{1}{2}\gamma \leq \pi \ , \qquad 0 \leq a < |z| < 2^{\gamma/\pi} r \ ,$$

and let $\varphi(\rho e^{i\theta})$ *be the harmonic measure of the arc* $|z| = a$, $\psi(\rho e^{i\theta})$ *that of the arc* $|z| = 2^{\gamma/\pi} r$ *relative to* D. *Then for* $2^{\gamma/\pi} a \leq \rho \leq r$,

$$\varphi(\rho) \geq \frac{1}{2}(a/\rho)^{\pi/\gamma}, \quad \psi(\rho e^{i\theta}) \leq 2(\rho/2r)^{\pi/\gamma} \ .$$

To prove these results it is again sufficient to consider the case $\gamma = \pi$. One easily finds that for the rotated domain $0 < \arg z < \pi$, $a < |z| < R$,

$$\varphi(\rho e^{i\theta}) = \sum_{n=1,3,\ldots} \frac{(a/\rho)^n - (a\rho/R^2)^n}{1 - a^{2n}/R^{2n}} \frac{4}{\pi n} \sin n\theta \quad ,$$

$$\psi(\rho e^{i\theta}) = \sum_{n=1,3,\ldots} \frac{(\rho/R)^n - (a^2/\rho R)^n}{1 - a^{2n}/R^{2n}} \frac{4}{\pi n} \sin n\theta \quad .$$

One then estimates the first term for $\theta^* = \frac{1}{2}\pi$, as well as the remainder after the first term.

5. Estimates for $\log|S|$ (general case). We follow the outline given in Section 2, but use the polynomial S given by (3.7) and the polynomial T derived from it by (2.6). We begin with certain

Preliminary estimates for $\log|S|$. By (3.10),

$$|S(e^{\sigma+i\theta})| \leq \sum_{0}^{\infty} \exp\{-\tfrac{1}{4}\lambda k^2 + (\sigma+1)k\}$$

$$= \exp\{(\sigma+1)^2/\lambda\} \sum_{0}^{\infty} \exp\{-\tfrac{1}{4}\lambda(k-2(\sigma+1)/\lambda)^2\}$$

$$\leq \exp\{(\sigma+1)^2/\lambda\} \cdot \{1 + \int_{-\infty}^{\infty} \exp(-\tfrac{1}{4}\lambda x^2)\,dx\}$$

$$= \{1+2(\pi/\lambda)^{\frac{1}{2}}\} \exp\{(\sigma+1)^2/\lambda\} .$$

Thus, remembering that $\lambda = 1/p \leq 1$, and taking $\sigma \geq 1$,

$$\log|S(z)| \leq p(\sigma+2)^2 \quad \text{for} \quad |z| \leq e^{\sigma} . \tag{5.1}$$

We now take $|z| < 1$. Then by (3.9)

$$|S(z)-1| < |z| + |z|^2 + \dots = \frac{|z|}{1-|z|} , \tag{5.2}$$

hence for $|z| \le \frac{1}{2}$

$$S(z) \neq 0 \quad \text{and} \quad |\arg S(z)| < \frac{1}{2}\pi \ .$$

<u>Preliminary estimates for</u> $\log |T|$. Since the z_j in (2.6) have absolute value $< R$ the maximum of $|T|$ on the circle $|z| = R+1$ is bounded by that of $|S|$. Thus by the maximum modulus theorem and (5.1),

$$\log|T(z)| \le p \log^2(8R) \quad \text{for} \quad |z| \le R+1 \ , \tag{5.3}$$

provided $R \ge 100$, say.

We next take $|z| \le 1/3$. Since $\frac{1}{2} < |z_j| < R$ we obtain from (2.6) and (5.2) that

$$\begin{cases} \frac{1}{2}(R+1)^{-N} \le |T(z)| \le 2 \cdot 6^N \ , \\ \\ |\arg T(z) - \arg T(1/3)| < \pi + N\pi \ . \end{cases} \tag{5.4}$$

By $\log T(z)$ we will denote the holomorphic branch of the logarithm, throughout the disc $|z| < \frac{1}{2}$ and the sector $|\arg z| < \frac{1}{2}\delta$, $0 < |z| < R$, which has imaginary part between $-\pi$ and π at $z = 1/3$. Then by (5.4)

$$\begin{cases} |\log T(z)| < \log 2 + N \log(R+1) + 2\pi + N\pi \\ \\ \qquad < 7 + 2N \log R \quad \text{for} \quad |z| \le 1/3 \ . \end{cases} \tag{5.5}$$

We now apply Lemma 2 to $\log T$ in our sector, taking $a = 1/3$. It is never a restriction to assume that $\delta \leq \pi$; we can then take $r = \frac{1}{2} R$. Thus by (5.5) and (5.3),

$$(5.6) \quad \begin{cases} |\log T(z)| < (7 + 2N \log R) \\ \\ + 8\{p \log^2(8R) + (7 + 2N \log R)\}(2R)^{\pi/\delta} \\ \\ < (p + N)(2R)^{2\pi/\delta} \end{cases}$$

throughout the region $|\arg z| \leq \frac{1}{4}\delta$, $1/3 \leq |z| \leq \frac{1}{2} R$; in the last step of (5.6) we have assumed $R \geq 10^3$, say.

<u>Estimates for</u> $\log|T(x)|$ <u>and</u> $\log|S(x)|$. We are now ready to use Lemma 3 to estimate $\log|\log T|$. We let D be the domain $|\arg z| < \frac{1}{4}\delta$, $1/3 < |z| < 2^{\delta/2\pi} R/4 < \frac{1}{2} R$. Taking $2/3 \leq x \leq R/4$ Lemma 3 shows that certainly

$$(5.7) \qquad \varphi(x) > \omega = (2R)^{-7/\delta} .$$

It thus follows from (5.5) and (5.6) that

$$\log|\log T(x)| \leq \omega \log(7 + 2N \log R) + (1-\omega) \log\{(p + N)(2R)^{2\pi/\delta}\} .$$

We will of course use (2.6) to estimate $S(x)$. Noting that $|x - z_j| \leq 2R$ a short computation shows that for $2/3 \leq x \leq R/4$

$$(5.8) \quad \begin{cases} \log|S(x)| \leq \log|T(x)| + N \log 2R \\ \\ \leq (N^{\omega} + 1)(p + N)^{1-\omega}(2R)^{3\pi/\delta} \end{cases},$$

where ω is given by (5.7). By (5.2) the answer holds also for $0 < x < 2/3$.

Final estimate for $\log|S|$. We again use Lemma 3, now to estimate $\log|S|$. We take D to be the domain $0 < \arg z < 2\pi$, $0 < |z| < R/4$. Taking $0 \leq \rho \leq R/16$ Lemma 3 shows that for all θ

$$\psi(\rho e^{i\theta}) \leq 2(8\rho/R)^{\frac{1}{2}} .$$

Thus, estimating the contributions of the real segment $(0, R/4)$ and the arc $|z| = R/4$ by (5.8) and (5.1), respectively, we conclude that

$$(5.9) \quad \begin{cases} \log|S(\rho e^{i\theta})| \leq (N^{\omega} + 1)(p + N)^{1-\omega}(2R)^{3\pi/\delta} \\ \\ \qquad + 2(8\rho/R)^{\frac{1}{2}}\, p \log^2(2R) , \end{cases}$$

with ω given by (5.7), and $R \geq 10^3$ as well as $\geq 16\rho$.

6. Conclusion of the proof (general case). We can now complete the proof of our

Theorem. Suppose that the partial sums s_n of a power series $\sum a_n z^n$ have the following property. For every n there is an angle of fixed opening $\delta > 0$ and vertex O in which s_n has at most An^α zeros ($0 \leq \alpha < 1$; we take $A \geq 1$ when $\alpha = 0$). Then

(6.1) $$a_n = O\{\exp(-\epsilon n^{2-\alpha})\},$$

with

(6.2) $$\epsilon \geq \frac{1}{\Omega A}, \quad \log_{10} \Omega = 10^{100/\delta}.$$

Furthermore, if k corresponds to a vertex of the Newton polygon of the power series and is sufficiently large, then there is another vertex index $k-p$ with $0 < p < \Omega A k^\alpha$.

Final step in the proof. By Cauchy's formula

$$|b_j \rho^j| \leq \max_\theta |S(\rho e^{i\theta})|,$$

hence the estimate for $\log |S|$ in (5.9) provides an upper bound in particular for $\log |b_p \rho^p|$. We compare this upper bound with the lower bound that follows from (3.10). Collecting the linear terms in p on the left hand side we obtain the inequality

$$(6.3)\quad \begin{cases} \{-1 + \log \rho - 2(8\rho/R)^{\frac{1}{2}} \log^2(2R)\}\, p \\ \\ \leq (N^{\omega} + 1)(p + N)^{1-\omega}(2R)^{3\pi/\delta} \ , \end{cases}$$

with ω given by (5.7) and $R \geq 10^3$, $R \geq 16\rho$.

We observe that if $\rho > e$ one can always choose R so large that the coefficient of p comes out positive; we do not want R too large, of course. The choice

$$2R = 10^8 \ , \quad \rho = e^4$$

makes the coefficient of p greater than $\frac{1}{2}$. Hence

$$(6.4)\qquad p \leq 2(N^{\omega} + 1)(p + N)^{1-\omega}\, 10^{24\pi/\delta} \ ,$$

where $\omega = 10^{-56/\delta}$. Thus either $p \leq N$, or else $N < p$ and then $p + N < 2p$, hence by a short computation

$$(6.5)\qquad p_n = p < \Omega^* \max(N, 1) \ , \ \Omega^* = 10^{25\pi/\delta\omega} \ .$$

We have $N \leq An^{\alpha}$ where $A \geq 1$ if $\alpha = 0$, hence $\max(N, 1) \leq An^{\alpha}$ provided n is chosen sufficiently large. Thus by (3.4), taking the special vertex index n greater than the vertex index k,

$$
(6.6)\quad \begin{cases} p_k = k^\alpha \mu_k \leq (3/2)k^\alpha \mu_n = (3/2)k^\alpha p_n/n^\alpha \\ \qquad < (3/2)\,\Omega^* A k^\alpha \end{cases}
$$

for all vertex indices $k \geq n_0$.

Lemma 1 finally shows that for all integers $k \geq k_1$

$$
(6.7)\qquad \log|1/a_k| = f(k) \geq g(k) \geq \epsilon\, k^{2-\alpha} \quad ,
$$

where

$$
(6.8)\qquad \epsilon = \frac{1}{9\Omega^* A} \quad , \quad \log_{10}\Omega^* = \frac{25\pi}{\delta}\, 10^{56/\delta} \quad .
$$

It is not hard to see that $9\Omega^*$ is bounded by the Ω given in (6.2) .

If k is a vertex index then so is $k - p_k$; by (6.6) we have $0 < p_k < \Omega A k^\alpha$ provided $k \geq n_0$.

REFERENCES

1. F. Carlson, Sur les fonctions entières, C. R., 179 (1924), 1583-1585.

2. __________, Sur les fonctions entières, Ark. Mat. Astr. Fys. 35 A (14)(1948), 1-18.

3. A. Edrei, Power series having partial sums with zeros in a half-plane, Proc. Amer. Math. Soc. 9 (1958), 320-324.

4. T. Ganelius, The zeros of the partial sums of power series, to appear in Duke Math. J.

5. G. W. Hedstrom and J. Korevaar, The zeros of the partial sums of certain small entire functions, to appear in Duke Math. J.

6. R. Jentzsch, Untersuchungen zur Theorie der Folgen analytischer Funktionen, Acta Math. 41 (1918), 219-251.

7. J. Korevaar, The zeros of approximating polynomials and the canonical representation of an entire function, Duke Math. J. 18 (1951), 573-592.

8. T. L. McCoy, Entire functions with restraints on the zeros of the partial sums, Ph. D. thesis, Univ. of Wisconsin, 1961.

9. P. Rosenbloom, On sequences of polynomials, especially sections of power series, Ph. D. thesis, Stanford Univ., 1944.

10. __________, Distribution of zeros of polynomials, Lectures on functions of a complex variable, Univ. of Michigan, Ann Arbor, 1955, 265-285.

11. E. C. Titchmarsh, The theory of functions, second edition, Oxford, 1947.